

# NONCOMPLEX SMOOTH 4-MANIFOLDS WITH LEFSCHETZ FIBRATIONS

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## 1. INTRODUCTION

Recently, B. Ozbagci and A. Stipsicz [12] proved that there are infinitely many pairwise nonhomeomorphic 4-manifolds admitting genus-2 Lefschetz fibration over  $S^2$  but not carrying any complex structure with either orientation. (For the definition of Lefschetz fibration, see [6].) Their result depends on a relation in the mapping class group of a closed orientable surface of genus 2. This relation with eight right Dehn twists was discovered by Y. Matsumoto [9] by a computer calculation, and it is the global monodromy of a Lefschetz fibration  $T^2 \times S^2 \# 4\overline{CP}^2 \rightarrow S^2$ , where  $S^2$  is the 2-sphere and  $T^2$  is the 2-torus.

In this paper, we generalize Matsumoto's relation to higher genus orientable surfaces. We find a relation involving  $2g + 4$  (resp.,  $2g + 10$ ) Dehn twists when the genus of the surface is even (resp., odd). Following the method of Ozbagci and Stipsicz, for every positive integer  $n$ , we obtain a 4-manifold  $X_n$  admitting a genus- $g$  Lefschetz fibration such that the fundamental group of  $X_n$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_n$  for every  $g \geq 2$ . We then deduce that the 4-manifold  $X_n$  does not admit any complex structure. This is the main result of this paper.

**Theorem 1.1.** *For every  $g \geq 2$ , there are infinitely many pairwise nonhomeomorphic 4-manifolds that admit genus- $g$  Lefschetz fibrations over  $S^2$  but do not carry any complex structure with either orientation.*

Our relation in the mapping class group given by Theorem 3.4 also shows that the minimal number of singular fibers in a nontrivial genus- $g$  Lefschetz fibration over  $S^2$  is less than or equal to  $2g + 4$  (resp.,  $2g + 10$ ) if  $g$  is even (resp., odd). This result was also obtained independently by C. Cadavid [3]. By definition, a Lefschetz fibration is nontrivial if it admits singular fibers. Stipsicz proved in [14] that this minimal

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*Date:* February 1, 2008.

1991 *Mathematics Subject Classification.* Primary 57N13, 57N05; Secondary 57R17, 20F38, 20F36.

*Key words and phrases.* 4-manifolds, Lefschetz fibrations, Mapping class groups.

number is in fact  $2g + 4$  (resp.,  $2g + 10$ ) if  $g$  is even (resp., odd) and greater than or equal to 6 (resp., greater than or equal to 15) among all 4-manifolds with  $b_2^+ = 1$ . See [7] and [13] for the other results related to this minimal number.

Here is how we obtain our relation in the mapping class group. Let  $\Sigma_g$  be a closed connected orientable surface of genus  $g$ . The hyperelliptic mapping class group of  $\Sigma_g$  is a quotient of the braid group  $B_{2g+2}$  on  $2g + 2$  strings. The quotient of the hyperelliptic mapping class group with the cyclic subgroup of order 2 generated by the hyperelliptic involution is isomorphic to the mapping class group of a sphere with  $2g + 2$  holes. The hyperelliptic mapping class group is equal to the mapping class group when  $g = 2$ . Using these facts, we lift Matsumoto's relation to the braid group  $B_6$  and generalize it to a relation in  $B_{2g+2}$ , although we do not say so explicitly. We then project it to the surface  $\Sigma_g$  to get our relation in the mapping class group of  $\Sigma_g$ .

For each positive integer  $n$ , by considering a product of conjugates of our relation with appropriate mapping classes, we obtain a relation in the mapping class group of  $\Sigma_g$  so that the fundamental group of the corresponding symplectic 4-manifold  $X_n$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_n$ . It follows from [12, proof of Theorem 1.3] that a symplectic manifold with fundamental group  $\mathbb{Z} \oplus \mathbb{Z}_n$  admits no complex structures.

In the last section, we determine the diffeomorphism type of the 4-manifold  $X$  admitting a genus- $g$  Lefschetz fibration over  $S^2$  corresponding to our relation given by Theorem 3.4. If  $g$  is even, then  $X$  is diffeomorphic to  $\Sigma_{g/2} \times S^2 \# \overline{4CP^2}$ . For this, we use a result of Stipsicz asserting that the only 4-manifold with  $b_2^+ = 1$  which admits a (relatively minimal) Lefschetz fibration with  $2g + 4$  vanishing cycles is  $\Sigma_{g/2} \times S^2 \# \overline{4CP^2}$ .

## 2. BRAID GROUPS

The braid group  $B_{2g+2}$  on  $2g + 2$  strings admits a presentation with generators  $\sigma_1, \sigma_2, \dots, \sigma_{2g+1}$  and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ if } 1 \leq i \leq 2g.$$

The subgroup of  $B_{2g+2}$  generated by  $\sigma_1, \sigma_2, \dots, \sigma_h$  is isomorphic to  $B_{h+1}$ . We identify this subgroup with  $B_{h+1}$ .

In the group  $B_{2g+2}$ , let us define the words  $\Delta_k = \sigma_1 \sigma_2 \cdots \sigma_k$  and  $\bar{\Delta}_k = \sigma_k \cdots \sigma_2 \sigma_1$  for each  $k = 1, 2, \dots, 2g + 1$ . We define  $\Delta_0 = 1$  for

the convention. For each  $k = 0, 1, \dots, g$ , let us also define

$$\beta_k = \bar{\Delta}_k \Delta_{2g+1-k} \Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1}$$

and

$$\beta = \bar{\Delta}_g^{g+1}.$$

We have, for example,  $\beta_0 = \Delta_{2g+1} \Delta_{2g}^{-1}$ ,  $\beta_1 = \bar{\Delta}_1 \Delta_{2g} \Delta_{2g-1}^{-1} \bar{\Delta}_1^{-1}$ , and  $\beta_g = \bar{\Delta}_g \Delta_{g+1} \Delta_g^{-1} \bar{\Delta}_g^{-1}$ . Note that  $\beta_k$  is the conjugate of  $\sigma_{2g+1-k}$  with the element  $\bar{\Delta}_k \Delta_{2g-k}$ .

The following lemma follows easily from the defining relations of the braid group.

**Lemma 2.1.** *The following relations hold in the group  $B_{2g+2}$ .*

- (a)  $\sigma_k \Delta_m = \Delta_m \sigma_{k-1}$  and  $\sigma_k^{-1} \Delta_m = \Delta_m \sigma_{k-1}^{-1}$  if  $1 < k \leq m$ ;
- (b)  $\sigma_k \bar{\Delta}_m = \bar{\Delta}_m \sigma_{k+1}$  and  $\sigma_k^{-1} \bar{\Delta}_m = \bar{\Delta}_m \sigma_{k+1}^{-1}$  if  $1 \leq k < m$ ;
- (c)  $\sigma_k \Delta_m = \Delta_m \sigma_k$  and  $\sigma_k \bar{\Delta}_m = \bar{\Delta}_m \sigma_k$  if  $k > m+1$ ; and
- (d)  $\Delta_g^k = \Delta_{g-1} \Delta_g^{k-1} \sigma_{g-k+1}$  and  $\bar{\Delta}_g^k = \sigma_{g-k+1} \bar{\Delta}_g^{k-1} \bar{\Delta}_{g-1}$  if  $1 \leq k \leq g$ .

**Lemma 2.2.** *In the braid group  $B_{2g+2}$ , we have the following:*

- (a) *The element  $\beta$  is equal to  $\bar{\Delta}_g \Delta_g \bar{\Delta}_{g-1}^g$ ; and*
- (b) *The element  $\bar{\Delta}_g \Delta_g$  is in the centralizer of  $B_g$ ; in particular, it commutes with  $\bar{\Delta}_{g-1}^g$ .*

*Proof.* We claim that  $\bar{\Delta}_g^g = \Delta_k \bar{\Delta}_g^{g-k} \bar{\Delta}_{g-1}^k$  for every  $0 \leq k \leq g$ . First of all, the claim holds trivially for  $k = 0$ . Suppose by induction that  $\bar{\Delta}_g^g = \Delta_k \bar{\Delta}_g^{g-k} \bar{\Delta}_{g-1}^k$ . By Lemma 2.1 (d), we have  $\bar{\Delta}_g^{g-k} = \sigma_{k+1} \bar{\Delta}_g^{g-k-1} \bar{\Delta}_{g-1}$ . Then we have

$$\begin{aligned} \bar{\Delta}_g^g &= \Delta_k \bar{\Delta}_g^{g-k} \bar{\Delta}_{g-1}^k \\ &= \Delta_k \sigma_{k+1} \bar{\Delta}_g^{g-k-1} \bar{\Delta}_{g-1} \bar{\Delta}_{g-1}^k \\ &= \Delta_{k+1} \bar{\Delta}_g^{g-(k+1)} \bar{\Delta}_{g-1}^{k+1}. \end{aligned}$$

In particular,  $\bar{\Delta}_g^g = \Delta_g \bar{\Delta}_{g-1}^g$ . The proof of (a) follows.

For  $k < g$ , by Lemma 2.1, we have  $\bar{\Delta}_g \Delta_g \sigma_k = \bar{\Delta}_g \sigma_{k+1} \Delta_g = \sigma_k \bar{\Delta}_g \Delta_g$ . Now the proof of (b) follows.  $\square$

**Lemma 2.3.** *For every  $0 \leq k \leq g-1$ , we have*

$$\Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1} \bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} = \bar{\Delta}_k \Delta_{2g-k} \gamma_k,$$

where  $\gamma_k = \bar{\Delta}_k \Delta_{2g-k-1} \Delta_{2g-k-2}^{-1} \bar{\Delta}_k^{-1}$ .

*Proof.* We use Lemma 2.1 several times:

$$\begin{aligned}
& \Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1} \bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \\
&= \sigma_{2g-k}^{-1} \cdots \sigma_{k+2}^{-1} \sigma_{k+1}^{-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} (\bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1}) \\
&= \sigma_{2g-k}^{-1} \cdots \sigma_{k+2}^{-1} \sigma_{k+1}^{-1} (\bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1}) \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \sigma_{2g-k}^{-1} \cdots \sigma_{k+3}^{-1} \sigma_{k+2}^{-1} \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \sigma_{2g-k}^{-1} \cdots \sigma_{k+3}^{-1} \sigma_{k+2}^{-1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \Delta_{2g-k} \sigma_{2g-k-1}^{-1} \cdots \sigma_{k+2}^{-1} \sigma_{k+1}^{-1} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \Delta_{2g-k} \sigma_{2g-k-1}^{-1} \cdots \sigma_{k+3}^{-1} \sigma_{k+2}^{-1} \bar{\Delta}_k \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_k \Delta_{2g-k-1} \sigma_{2g-k-2}^{-1} \cdots \sigma_{k+2}^{-1} \sigma_{k+1}^{-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_k \Delta_{2g-k-1} \Delta_{2g-k-2}^{-1} \bar{\Delta}_k^{-1} \\
&= \bar{\Delta}_k \Delta_{2g-k} \gamma_k.
\end{aligned}$$

□

The main result of this section is the next theorem.

**Theorem 2.4.** *In the braid group  $B_{2g+2}$ , we have the relation*

$$\beta_0 \beta_1 \beta_2 \cdots \beta_g \beta^2 = \Delta_{2g+1} \Delta_{2g} \cdots \Delta_3 \Delta_2 \Delta_1.$$

*Proof.* Recall that, for any  $h \leq 2g+2$ , we identify the group  $B_h$  with the subgroup of  $B_{2g+2}$  generated by the elements  $\sigma_1, \sigma_2, \dots, \sigma_{h-1}$ .

The proof of the theorem is by induction on  $g$ . Suppose that  $g = 0$ . In the group  $B_2$ ,  $\beta_0 = \Delta_1 \Delta_0^{-1}$  and  $\beta = \Delta_0$ . Thus  $\beta_0 \beta^2 = \Delta_1$ . Hence, the conclusion of the theorem holds for  $g = 0$ .

In the subgroup  $B_{2g}$  of  $B_{2g+2}$ , let us define

$$\gamma_k = \bar{\Delta}_k \Delta_{2g-1-k} \Delta_{2g-2-k}^{-1} \bar{\Delta}_k^{-1}, \quad 0 \leq k \leq g-1$$

and

$$\gamma = \bar{\Delta}_{g-1}^g.$$

Then, by the induction hypothesis

$$\gamma_0 \gamma_1 \gamma_2 \cdots \gamma_{g-1} \gamma^2 = \Delta_{2g-1} \Delta_{2g-2} \cdots \Delta_3 \Delta_2 \Delta_1.$$

Let us also define  $\gamma_g = 1$  for the convention.

In the group  $B_{2g+2}$ , we claim that

$$\beta_k \beta_{k+1} \cdots \beta_g \beta^2 = \bar{\Delta}_k \Delta_{2g+1-k} \bar{\Delta}_k \Delta_{2g-k} \gamma_k \gamma_{k+1} \cdots \gamma_{g-1} \gamma_g \gamma^2.$$

The proof of this claim is by induction on  $g - k$ . We start with the following computation:

$$\begin{aligned}
 \beta_g \beta^2 &= \bar{\Delta}_g \Delta_{g+1} \Delta_g^{-1} \bar{\Delta}_g^{-1} (\bar{\Delta}_g \Delta_g \bar{\Delta}_{g-1}^g)^2 \\
 &= \bar{\Delta}_g \Delta_{g+1} \bar{\Delta}_{g-1}^g \bar{\Delta}_g \Delta_g \bar{\Delta}_{g-1}^g \\
 &= \bar{\Delta}_g \Delta_{g+1} \bar{\Delta}_g \Delta_g (\bar{\Delta}_{g-1}^g)^2 \\
 &= \bar{\Delta}_g \Delta_{g+1} \bar{\Delta}_g \Delta_g \gamma_g \gamma^2.
 \end{aligned}$$

Hence, the claim holds for  $k = g$ . Suppose inductively that

$$\beta_{k+1} \beta_{k+2} \cdots \beta_g \beta^2 = \bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \gamma_{k+1} \gamma_{k+2} \cdots \gamma_g \gamma^2.$$

Then by Lemma 2.3 we get

$$\begin{aligned}
 \beta_k \beta_{k+1} \cdots \beta_g \beta^2 &= \bar{\Delta}_k \Delta_{2g+1-k} (\Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1} \bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1}) \gamma_{k+1} \gamma_{k+2} \cdots \gamma_g \gamma^2 \\
 &= \bar{\Delta}_k \Delta_{2g+1-k} \bar{\Delta}_k \Delta_{2g-k} \gamma_k \gamma_{k+1} \cdots \gamma_g \gamma^2.
 \end{aligned}$$

Hence, the claim is proved. For  $k = 0$ , in particular, we obtain

$$\begin{aligned}
 \beta_0 \beta_1 \beta_2 \cdots \beta_g \beta^2 &= \bar{\Delta}_0 \Delta_{2g+1} \bar{\Delta}_0 \Delta_{2g} \gamma_0 \gamma_1 \cdots \gamma_g \gamma^2 \\
 &= \Delta_{2g+1} \Delta_{2g} \gamma_0 \gamma_1 \cdots \gamma_g \gamma^2 \\
 &= \Delta_{2g+1} \Delta_{2g} \Delta_{2g-1} \cdots \Delta_3 \Delta_2 \Delta_1.
 \end{aligned}$$

This finishes the proof of the theorem.  $\square$

### 3. MAPPING CLASS GROUPS

Let  $\Sigma_g$  be a closed connected orientable surface of genus  $g$  embedded in  $\mathbb{R}^3$  such that it is invariant under the involution  $J(x, y, z) = (-x, y, -z)$  (cf. Fig. 4). Notice that  $J$  is the rotation about  $y$ -axis by  $\pi$ . We orient  $\Sigma_g$  so that the unit normal vectors are pointing outward. Let  $j$  be the isotopy class of  $J$ . Let us denote by  $M_g$  the mapping class group of  $\Sigma_g$  which is the group of isotopy classes of orientation-preserving diffeomorphisms of  $\Sigma_g$ . The hyperelliptic mapping class group is defined as the centralizer  $C_{M_g}(j)$  of  $j$  in  $M_g$ , the subgroup consisting of those the mapping classes that commute with  $j$ .

Throughout this paper, we use functional notation. That is, for any two mapping classes  $f$  and  $g$ , the multiplication  $fg$  means that  $g$  is applied first.

For a simple closed curve  $a$  on the oriented surface  $\Sigma_g$ , by the abuse of notation, a right Dehn twist about  $a$  and its isotopy class is denoted by  $t_a$ .

Let us consider the simple closed curves  $A_1, A_2, \dots, A_{2g+1}$  on  $\Sigma_g$  defined as follows:  $A_{2k} = b_k$ ,  $A_1 = a_1$ ,  $A_{2k-1} = a_k a_{k+1}^{-1}$ , and  $A_{2g+1} = a_g$ ,

where  $a_k$  and  $b_k$  are the curves shown in Fig. 4. Let  $t_k$  denote the right Dehn twist about  $A_k$ . The simple closed curves  $A_k$  are invariant under  $J$ . It follows that Dehn twists  $t_1, t_2, \dots, t_{2g+1}$  commute with  $J$ . Hence, they are contained in the hyperelliptic mapping class group.

The involution  $J$  has  $2g + 2$  fixed points. Hence, we have a branched covering  $p : \Sigma_g \rightarrow S^2$  branching over  $2g + 2$  points. Let us mark these  $2g + 2$  points on  $S^2$ , and let  $\Sigma_{0,2g+2}$  be the resulting surface. Notice that the interior of each  $p(A_k)$  is an embedded arc on  $\Sigma_{0,2g+2}$  connecting two distinct marked points and that it is disjoint from  $p(A_l)$  for  $k \neq l$ . Let us denote by  $w_k$  the isotopy class of a right half twist about  $p(A_k)$ . Thus, if we orient the arc  $p(A_k)$  arbitrarily, then  $w_k(p(A_k))$  (defined up to isotopy) is isotopic to the arc  $p(A_k)^{-1}$ . Therefore,  $w_k^2$  is the right Dehn twist about the boundary component of a regular neighborhood of  $p(A_k)$ . It is well-known that the half twists  $w_1, w_2, \dots, w_{2g+1}$  generate the mapping class group  $M_{0,2g+2}$  of  $\Sigma_{0,2g+2}$  (cf. [1, Theorem 4.5]). Here, the group  $M_{0,2g+2}$  is defined to be the group of the isotopy classes of the orientation-preserving diffeomorphisms of  $\Sigma_{0,2g+2}$  that preserve the marked points setwise. The isotopies are assumed to fix each marked point.

**Theorem 3.1.** *The hyperelliptic mapping class group  $C_{M_g}(J)$  is generated by the Dehn twists  $t_1, t_2, \dots, t_{2g+1}$ , the function given by  $\Psi(t_k) = w_k$  on the generators defines a surjective homomorphism*

$$\Psi : C_{M_g}(J) \rightarrow M_{0,2g+2},$$

*and the kernel of  $\Psi$  is  $\langle J \rangle$ , which is a subgroup of order 2.*

Theorem 3.1 was proved by J. Birman and H. Hilden [2]. They also obtained a presentation of the hyperelliptic mapping class group. Since  $t_i t_j = t_j t_i$  for  $|i - j| \geq 2$  and  $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$  in the group  $C_{M_g}(J)$ , the fact that  $t_1, t_2, \dots, t_{2g+1}$  generate  $C_{M_g}(J)$  implies that  $\sigma_k \mapsto t_k$  defines a surjective homomorphism  $B_{2g+2} \rightarrow C_{M_g}(J)$ .

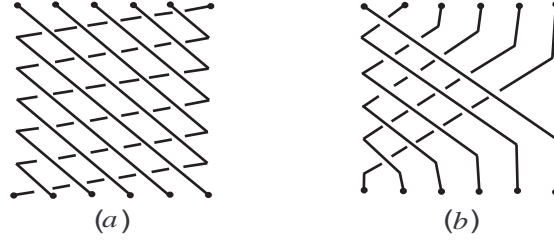
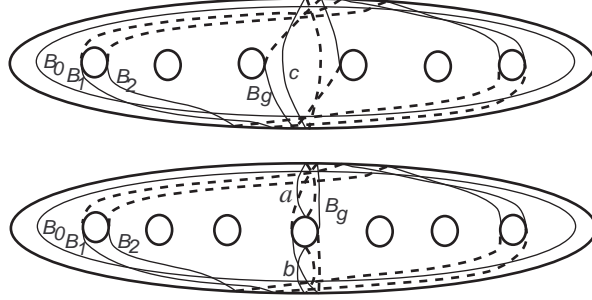
The following lemma is easy to prove (cf. Fig. 1 (a)).

**Lemma 3.2.** *In the group  $M_{0,2g+2}$ , the element  $(w_k \cdots w_{2g+1})^{k+1}$  is equal to the right Dehn twist about the boundary component of a regular neighborhood of  $p(A_1) \cup p(A_2) \cup \cdots \cup p(A_k)$ .*

In order to state the main result of this section, let us consider the simple closed curves  $B_k, a, b$ , and  $c$  illustrated in Fig. 2. Note that  $a$  and  $b$  are defined for odd  $g$ , and  $c$  is defined for even  $g$ .

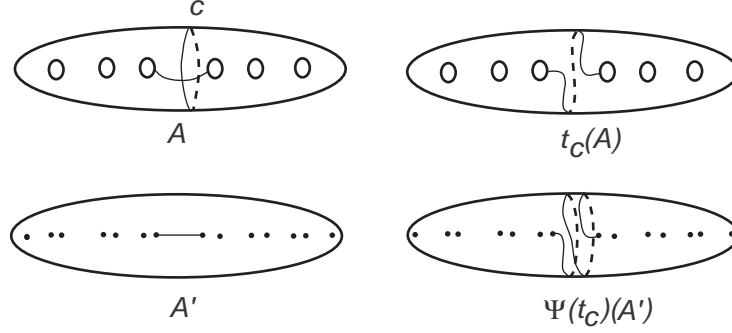
**Lemma 3.3.** *The following relations hold in the mapping class group:*

- (a)  $t_c = (t_g \cdots t_2 t_1)^{2(g+1)}$  if  $g$  is even; and
- (b)  $t_a t_b = (t_g \cdots t_2 t_1)^{g+1}$  if  $g$  is odd.

FIGURE 1. The words  $(\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1)^6$  and  $\Delta$  in  $B_6$ .FIGURE 2. The simple closed curves  $B_k$ ,  $a$ ,  $b$  and  $c$ .

*Proof.* Suppose that  $g$  is even. Consider the branched covering  $p : \Sigma_g \rightarrow S^2$ . Notice that  $p(c)$  is a simple closed curve on  $\Sigma_{0,2g+2}$ . The projection  $\Psi(t_c)$  of  $t_c$  to  $\Sigma_{0,2g+2}$  is the square of the Dehn twist about  $p(c)$ . This can be seen geometrically as follows. Consider the arcs  $p(A_1), p(A_2), \dots, p(A_{2g+1})$ . Since the surface obtained by cutting  $\Sigma_{0,2g+2}$  along these arcs is a disc without any marked points in the interior, in order to show that  $\Psi(t_c) = t_{p(c)}^2$ , it is enough to check that the actions of  $\Psi(t_c)$  and  $t_{p(c)}^2$  on these arcs are the same (up to isotopy). To see the action of  $\Psi(t_c)$  on an arc  $A'$ , lift  $A'$  to  $\Sigma_g$ , apply  $t_c$ , and then project it down to  $\Sigma_{0,2g+2}$  (cf. Fig. 3). Since  $t_{p(c)} = (w_g \cdots w_2 w_1)^{g+1}$  by Lemma 3.2, we conclude that  $\Psi(t_c) = \Psi((t_g \cdots t_2 t_1)^{2(g+1)})$ . Hence,  $(t_g \cdots t_2 t_1)^{2(g+1)}$  is equal to either  $t_c$  or  $jt_c$ . We rule out the latter possibility as follows. It is easily checked that  $(t_g \cdots t_2 t_1)^{2(g+1)}$  acts trivially on the first homology group  $H_1(\Sigma_g; \mathbb{Z})$ . The action of  $t_c$  is also trivial since  $c$  is null homologous. On the other hand,  $j$  acts as the minus identity on  $H_1(\Sigma_g; \mathbb{Z})$ .

The part (b) is proved similarly. □

FIGURE 3. Projection of the Dehn twist  $t_c$ .

**Theorem 3.4.** *In the mapping class group  $M_g$ , the following relations between right Dehn twists hold:*

- (a)  $(t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_c)^2 = 1$  if  $g$  is even;
- (b)  $(t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_a^2t_b^2)^2 = 1$  if  $g$  is odd.

*Proof.* In the mapping class group  $M_g$  of  $\Sigma_g$ , for each  $k = 0, 1, 2, \dots, g$ , we define  $\Delta_k, \bar{\Delta}_k, \beta_k$ , and  $\beta$  as in Section 2 by replacing  $\sigma_i$  by  $t_i$ . Recall that  $t_i$  is the (right) Dehn twist about the simple closed curve  $A_i$ . Hence,

$$\beta_k = (\bar{\Delta}_k \Delta_{2g-k}) t_{2g+1-k} (\bar{\Delta}_k \Delta_{2g-k})^{-1}.$$

It is easy to see that  $\bar{\Delta}_i \Delta_{2g-i}(A_{2g+1-i}) = B_i$ . Since  $ft_e f^{-1} = t_{f(e)}$  for any  $f \in M_g$  and for any simple closed curve  $e$ , we conclude that  $\beta_k = t_{B_k}$ . Also, by Lemma 3.3,  $\beta^2 = t_c$  if  $g$  is even and  $\beta^2 = t_a^2 t_b^2$  if  $g$  is odd. Let us define the word

$$W = \begin{cases} (t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_c)^2 & \text{if } g \text{ is even,} \\ (t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_a^2t_b^2)^2 & \text{if } g \text{ is odd.} \end{cases}$$

Hence,  $W = (\beta_0\beta_1\cdots\beta_g\beta^2)^2$ . Let  $\Delta = \Delta_{2g+1}\Delta_{2g}\cdots\Delta_2\Delta_1$ . Since  $t_i t_j = t_j t_i$  for  $|i-j| \geq 2$  and  $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$  by Theorem 2.4, we obtain  $W = \Delta^2$ . Now the element  $\Psi(\Delta)$  is of order 2 in  $M_{0,2g+2}$  (cf. Fig. 1 (b) for  $g = 6$ ), where  $\Psi$  is the epimorphism in Theorem 3.1. Hence, either  $\Delta^2 = 1$  or  $\Delta^2 = j$ . It is easy to verify that  $\Delta^2$  acts trivially on the first homology group  $H_1(\Sigma_g; \mathbf{Z})$  but  $j$  does not. Therefore,  $W = 1$ .

This finishes the proof of the theorem.  $\square$



4. NONCOMPLEX GENUS- $g$  LEFSCHETZ FIBRATIONS

In this section, we prove the main result of this paper. We assume from now on that  $g \geq 2$ . We construct a 4-manifold  $X_n$  admitting genus- $g$  Lefschetz fibration with fundamental group  $\mathbb{Z} \oplus \mathbb{Z}_n$  for every positive integer  $n$ . Then we conclude the main result of this paper from the proof of the main result of [12]. As the model for a closed connected oriented surface  $\Sigma_g$ , we will consider the one embedded in  $\mathbb{R}^3$  as shown in Fig. 4.

For any two elements  $x$  and  $y$  in a group, we denote  $xyx^{-1}$  and  $xyx^{-1}y^{-1}$  by  $x^y$  and  $[x, y]$ , respectively.

Let us consider the word  $W$  in  $M_g$  defined by

$$W = \begin{cases} (t_{B_0}t_{B_1}t_{B_2} \cdots t_{B_g}t_c)^2 & \text{if } g \text{ is even,} \\ (t_{B_0}t_{B_1}t_{B_2} \cdots t_{B_g}t_a^2t_b^2)^2 & \text{if } g \text{ is odd.} \end{cases}$$

By Theorem 3.4,  $W = 1$  in  $M_g$ . Since the conjugation of a right Dehn twist with an element of  $M_g$  is again a right Dehn twist, the word  $W^f$  is a product of right Dehn twists for any mapping class  $f$ . For each positive integer  $n$ , we define

$$W_n = \begin{cases} WW^{t_{a_1}}W^{t_{a_2}} \cdots W^{t_{a_{r-1}}}W^{t_{a_r}^n}W^{t_{b_{r+2}}}W^{t_{b_{r+3}}} \cdots W^{t_{b_g}} & \text{if } g = 2r, \\ WW^{t_3^n}W^{t_5}W^{t_7} \cdots W^{t_{2r+1}}W^{t_{b_{r+2}}}W^{t_{b_{r+3}}} \cdots W^{t_{b_g}} & \text{if } g = 2r + 1, \end{cases}$$

where  $a_i$  and  $b_i$  are simple closed curves given in Fig. 4 (considered up to isotopy). Note that  $W_n$  is a product of  $g(2g+4)$  (resp.,  $g(2g+10)$ ) right Dehn twists if  $g$  is even (odd).

The word  $W_n$  is equal to the identity in the mapping class group  $M_g$ . Let  $X$  and  $X_n$  be the smooth 4-manifolds that admit the genus- $g$  Lefschetz fibrations over  $S^2$  whose global monodromies are  $W$  and  $W_n$ , respectively. Thus,  $X_n$  is the fiber sum of  $g$  copies of  $X$ .

**Theorem 4.1.** *The fundamental group  $\pi_1(X_n)$  of  $X_n$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_n$ .*

*Proof.* Let  $a_k$  and  $b_k$  be the standard generators of  $\pi_1(\Sigma_g)$  illustrated in Fig. 4. By the theory of Lefschetz fibrations,  $\pi_1(X_n)$  is isomorphic to the quotient of  $\pi_1(\Sigma_g)$  by the normal subgroup generated by the vanishing cycles.

Suppose first that  $g = 2r$ . It is easy to check that up to conjugation the following equalities hold in  $\pi_1(\Sigma_g)$ :

- $B_0 = b_1b_2 \cdots b_g$ ;
- $B_{2k-1} = a_kb_kb_{k+1} \cdots b_{g+1-k}c_{g+1-k}a_{g+1-k}$ ,  $1 \leq k \leq r$ ;
- $B_{2k} = a_kb_{k+1}b_{k+2} \cdots b_{g-k}c_{g-k}a_{g+1-k}$ ,  $1 \leq k \leq r-1$ ;
- $B_g = B_{2r} = a_rc_ra_{r+1}$ ;
- $c = c_r = [a_1, b_1][a_2, b_2] \cdots [a_r, b_r]$ .

The vanishing cycles corresponding to  $W^{t_{a_k}}$  are the set

$$\{a_k B_0, \dots, a_k B_{2k-1}, B_{2k}, \dots, B_g, c\}, \quad 1 \leq k \leq r-1.$$

Similarly, the vanishing cycles corresponding to  $W^{t_{a_r}}$  and  $W^{t_{b_{g+1-k}}}$  are

$$\{a_r^n B_0, \dots, a_r^n B_{g-1}, B_g, c\}$$

and

$$\{B_0, \dots, B_{2k-2}, b_{g+1-k}^{-1} B_{2k-1}, b_{g+1-k}^{-1} B_{2k}, B_{2k+1}, \dots, B_g, c\}, \quad 1 \leq k \leq r-1,$$

respectively. It follows that the fundamental group of  $X_n$  has a presentation with generators  $a_1, b_1, a_2, b_2, \dots, a_g, b_g$  and relations

- $\Pi_{k=1}^g [a_k, b_k] = 1;$
- $B_0 = B_1 = B_2 = \dots = B_g = c = 1;$
- $a_1 = a_2 = \dots = a_{r-1} = a_r^n = b_{r+2} = b_{r+3} = \dots = b_g = 1.$

It is easy to see that this presentation is equivalent to the presentation with generators  $a_r, b_r$  and relations  $a_r^n = [a_r, b_r] = 1$ .

Suppose now that  $g = 2r + 1$ . A similar argument as in the case of even  $g$  shows that the fundamental group of  $X_n$  has a presentation with generators  $a_1, b_1, a_2, b_2, \dots, a_g, b_g$  and relations

- $\Pi_{k=1}^g [a_k, b_k] = 1;$
- $B_0 = B_1 = B_2 = \dots = B_g = a = b = 1;$
- $(a_2 a_1^{-1})^n = a_3 a_2^{-1} = a_4 a_3^{-1} = \dots = a_{r+1} a_r^{-1} = b_{r+2} = b_{r+3} = \dots = b_g = 1.$

Since  $a = a_{r+1}$ , this presentation is equivalent to the presentation with generators  $a_1, b_1$  and relations  $a_1^n = [a_1, b_1] = 1$ .

This completes the proof of the theorem.  $\square$

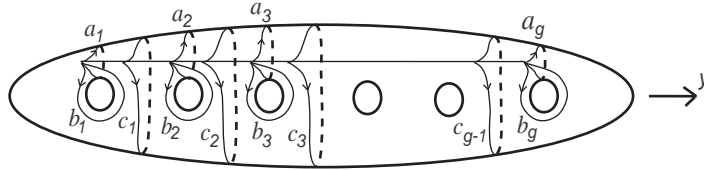


FIGURE 4. Generators of the fundamental group.

The following theorem can be concluded from [12, proof of Theorem 1.3].

**Theorem 4.2.** *Let  $M$  be an orientable 4-manifold such that  $b_2^+(M) \geq 1$  and  $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}_n$ . Then  $M$  does not carry any complex structure.*

*Proof of Theorem 1.1.* The smooth 4-manifold  $X_n$  admits a genus- $g$  Lefschetz fibration for every positive integer  $n$ . Since  $X_n$  is symplectic by a result of R. Gompf [6], we have  $b_2^+(X_n) \geq 1$ . We showed above that  $\pi_1(X_n) = \mathbb{Z} \oplus \mathbb{Z}_n$ . Hence, the manifold  $X_n$  is not homeomorphic to  $X_m$  for  $n \neq m$ . By Theorem 4.2, the manifold  $X_n$  and the manifold obtained from it by reversing the orientation do not admit any complex structure.  $\square$

## 5. THE 4-MANIFOLD ADMITTING A LEFSCHETZ FIBRATION WITH GLOBAL MONODROMY $W$ .

In this section, we determine the 4-manifold corresponding to the word  $W$  given in Section 3.

**The case of even  $g$ .** Notice that simple closed curves  $B_0, B_1, \dots, B_g$ , and  $c$  are invariant under the involution  $J$ . Hence, the genus- $g$  Lefschetz fibration  $X \rightarrow S^2$  with global monodromy  $W$  is hyperelliptic.

**Theorem 5.1** ([8],[9],[4],[11]). *Let  $M$  be a 4-manifold that admits a hyperelliptic Lefschetz fibration of genus  $g$  over  $S^2$ . Let  $m$  and  $s = \sum_{h=1}^{[g/2]} s_h$  be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively, where  $s_h$  denotes the number of separating vanishing cycles that separate the genus- $g$  surface into two surfaces one of which has genus  $h$ . Then the signature of  $M$  is*

$$\sigma(M) = -\frac{g+1}{2g+1}m + \sum_{h=1}^{[g/2]} \left( \frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

Since there are  $2g+4$  vanishing cycles, Euler characteristic of  $X$  is  $\chi(X) = 2(2-2g) + 2g+4 = 8-2g$ . There are only two separating vanishing cycles, and they bound a surface of genus  $g/2$  on both sides. Hence, the signature of  $X$  is

$$\sigma(X) = -\frac{g+1}{2g+1}(2g+2) + \left( \frac{4\frac{g}{2}(g-\frac{g}{2})}{2g+1} - 1 \right) 2 = -4.$$

The group  $\pi_1(X)$  has a presentation with generators  $a_1, b_1, \dots, a_g, b_g$  and relations  $\prod_{k=1}^g [a_k, b_k] = B_0 = B_1 = B_2 = \dots = B_g = c = 1$ . It is now easy to see that  $H_1(X; \mathbb{Z}) = \mathbb{Z}^g$ . In particular,  $b_1(X) = g$ . It follows from  $\chi(X) = 8-2g$  and  $\sigma(X) = -4$  that  $b_2^+(X) = 1$ . By [14, Remark 4.5(a)],  $X$  is diffeomorphic to  $\Sigma_{g/2} \times S^2 \# 4CP^2$  if  $g \geq 6$ .

**The case of odd  $g$ .** Suppose that  $g$  is at least 3 and odd. Let  $X \rightarrow S^2$  be the genus- $g$  Lefschetz fibration with global monodromy  $W$ . Since there are  $2g+10$  singular fibers, the Euler characteristic of  $X$  is  $\chi(X) = 2(2-2g) + 2g + 10 = 14 - 2g$ . The fundamental group  $\pi_1(X)$  of  $X$  has a presentation with generators  $a_1, b_1, \dots, a_g, b_g$  and relations  $\prod_{k=1}^g [a_k, b_k] = B_0 = B_1 = B_2 = \dots = B_g = a = b = 1$ . It is now easy to see that  $H_1(X; \mathbb{Z}) = \mathbb{Z}^{g-1}$ . In particular,  $b_1(X) = g - 1$ . Hence,

$$14 - 2g = 2 - 2b_1(X) + b_2(X) = 2 - 2(g - 1) + b_2(X);$$

that is,  $b_2(X) = 10$ .

The manifold  $X$  is symplectic. Since  $1 - b_1 + b_2^+$  is even for any symplectic manifold, we conclude that  $b_2^+(X)$  is odd and is between 1 and 9. Hence,  $b_2^-(X)$  is also odd and is between 1 and 9.

We now determine the signature of  $X$ . A handlebody decomposition for  $X$  is obtained as follows. Start with  $\Sigma_g \times D^2$ , where  $D^2$  is the 2-disc. Its boundary is  $\Sigma_g \times S^1$ . Attach a 2-handle along each vanishing cycle (by counting its multiplicity) with the  $-1$  framing relative to the product framing. The cores of the first two 2-handles attached along  $a$  gives us a  $(-2)$ -sphere  $S_1$ . Denote the class of  $S_1$  in  $H_2(X; \mathbb{R})$  by  $[S_1]$ . Similarly, the cores of the second and the third 2-handles, and the third and the fourth 2-handles give two  $(-2)$ -spheres  $S_2$  and  $S_3$ . Note that  $[S_1][S_3] = 0$ . Orient each  $S_i$  so that  $[S_1][S_2] = [S_2][S_3] = -1$ . Similarly, the four 2-handles glued along  $b$  give three  $(-2)$ -spheres  $S_4, S_5, S_6$  with  $[S_4][S_5] = [S_5][S_6] = -1$  and  $[S_4][S_6] = 0$ . Since  $a$  and  $b$  are disjoint,  $[S_i][S_j] = 0$  for  $1 \leq i \leq 3$  and  $4 \leq j \leq 6$ . The first handles attached along  $a$  and  $b$  give a surface  $S_7$  of genus  $(g-1)/2$  such that  $[S_7]^2 = -2$ ,  $[S_7][S_1] = [S_7][S_4] = -1$ , and  $[S_7][S_i] = 0$  for  $i = 2, 3, 5, 6$ .

The homology classes  $[S_1], \dots, [S_7]$  are linearly independent. Hence, they form a basis for a subspace  $V$  of  $H_2(X; \mathbb{R})$  of dimension 7. The matrix of the intersection form restricted to  $V$  in the above basis is the matrix  $-A$ , where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

It is easily check that the matrix  $A$  is positive definite. Hence, the restriction of the intersection form to  $V$  is negative definite.

On the other hand, we have  $[F] \neq 0$ ,  $[F]^2 = 0$  and  $[F] \in V^\perp$ , where  $F$  is a generic fiber and  $V^\perp$  is the orthogonal complement of  $V$ . Since the restriction of the intersection form to  $V^\perp$  is nondegenerate, there is a class  $[S_8] \in V^\perp$  with  $[S_8]^2 < 0$ . Thus, the restriction of the intersection form to the 8-dimensional subspace generated by  $[S_1], \dots, [S_8]$  is negative definite. Therefore,  $b_2^-(X)$  is at least 8, and hence  $b_2^-(X) = 9$ , since it is also odd. Consequently, we have  $b_2^+(X) = 1$  and  $\sigma(X) = -8$ .

**Remarks 5.2.** (1) Stipsicz [14] points out that when  $g$  is odd, the manifold  $X$  that admits a Lefschetz fibration over  $S^2$  with global monodromy  $W$  is diffeomorphic to  $\Sigma_{(g-1)/2} \times S^2 \# 8\overline{CP}^2$ .

(2) R. Fintushel and R. Stern [5] constructed infinite classes of simply connected homeomorphic but nondiffeomorphic symplectic manifolds, all of which admit Lefschetz fibrations of a fixed fiber genus.

**Acknowledgments.** I thank András I. Stipsicz and Sergey Finashin for helpful conversations and for suggestions in the computation of the signature in the case of odd  $g$ . I also thank Yildiray Ozan for answering numerous questions.

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